

Horizons of stability in matrix Hamiltonians

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Abstract

Non-Hermitian Hamiltonians $H \neq H^\dagger$ possess the real (i.e., observable) spectra inside certain specific, “physical” domains of parameters $\mathcal{D} = \mathcal{D}(H)$. In general, the determination of their “observability-horizon” boundaries $\partial\mathcal{D}$ is difficult. We list the pseudo-Hermitian real N by N matrix Hamiltonians for which the “prototype” horizons $\partial\mathcal{D}$ are defined by closed analytic formulae.

1 Introduction

In Landau's textbook [1] on Quantum Mechanics the emergence of an instability in a system is illustrated via a particle in the potential $V(\vec{x}) = G/|\vec{x}|^2$. The critical value $G_{(min)} = -1/4$ of its strength represents a “horizon” beyond which the particle starts falling on the center. *Vice versa*, the system remains stable and physical on all the interval $\mathcal{D} = (-1/4, \infty)$ of couplings G . From the pragmatic point of view the Landau's example is not too well selected since the falling particle should release, hypothetically, an infinite amount of energy during its fall. A better example of the loss of stability is provided by the Dirac's electron which moves in a superstrong Coulomb potential: In the language of physics, particle-antiparticle pairs are created in the system beyond a critical charge ($Z_{(max)} = 137$ in suitable units [2]).

What is shared by the above two sample Hamiltonians $H(\lambda)$ is that they are well defined in a certain domain \mathcal{D} of parameters λ while they lose sense and applicability for parameter(s) beyond certain horizon(s) $\lambda_{(max/min)}$. On a less intuitive level, similar situations have been studied by Kato [3]. He considered certain finite-matrix toy Hamiltonians $H(\lambda)$ in complex plane of λ and deduced that the related (in general, complex) spectra $E_n(\lambda)$ change smoothly with the variation of the parameter λ unless one encounters certain critical, “exceptional” points $\lambda^{(EP)}$.

Several recent microwave measurements [4] confirmed the observability of the abstract Kato's exceptional points $\lambda^{(EP)}$ in practice. These experiments re-attracted attention to the theoretical analyses of the EP horizons, say, in nuclear physics where many nuclei can, abruptly, lose their stability [5, 6]. The presence of EPs may be also detected in the random-matrix ensembles with various interpretations [7] and in optical systems (where EPs are called “degeneracies” [8]). In classical magnetohydrodynamics the Kato's exceptional points may even happen to lie *inside* the domain of acceptable parameters, separating merely the different dynamical regimes of the so called α^2 -dynamoes [9].

The obvious theoretical appeal of the problem of stability may be perceived as one of the explanations of the recent growth of popularity of the so called pseudo-Hermitian Hamiltonians in quantum physics [10]. Indeed, one of their distinctive features is that their spectra are real (i.e., observable) in parametric domains \mathcal{D} with, sometimes, very complicated and strongly Hamiltonian-dependent shape of their EP boundaries $\partial\mathcal{D}$. For an uninterrupted development of their study it was very fortunate that a successful semiquantitative description of the spiked-shaped

horizons $\partial\mathcal{D}$ has been found, by Dorey, Duncan and Tateo [11], for an important class of analytic (polynomial and power-law, often called \mathcal{PT} -symmetric) potentials $V(x, \lambda)$ with promising relevance in quantum field theory [12, 13].

In a few of our own recent studies of EPs in pseudo-Hermitian Hamiltonians H [14] - [21] we paid detailed attention to the possibilities of a deeper geometric understanding of the structure of the domains $\mathcal{D}(H)$ of quasi-Hermiticity (the name means that the spectrum remains real for parameters inside \mathcal{D} – cf. ref. [5] for an older, well written introduction of this concept). After we review some of the known results in section 2 we shall combine, in section 3, the methods of algebra (of solvable equations) and analysis (of elementary curves) in a new approach to the problem. In this way, the list of results of ref. [18] (based mainly on computer-assisted symbolic manipulations) and of ref. [19] (which used, predominantly, perturbation-expansion methods) will be complemented by a number of new non-perturbative items. They will be discussed and summarized in sections 4 and 5.

2 Matrix models

2.1 Inspiration: two-dimensional Hilbert space

The first nontrivial schematic illustration of the current Schrödinger's bound-state problem is provided by the two-by-two real-matrix model

$$H|\psi\rangle = E|\psi\rangle, \quad H = H(a, b, d) = \begin{pmatrix} a & b \\ b & d \end{pmatrix} = H^\dagger(a, b, d).$$

Its three-parametric spectrum is *always* real and, therefore, observable,

$$E = E_\pm(a, b, d) = \frac{1}{2} \left[a + d \pm \sqrt{(a - d)^2 + 4b^2} \right].$$

In the context of \mathcal{PT} -symmetric Quantum Mechanics [12, 22], the parallel two-by-two example is very similar

$$H = H'(a, b, d) = \begin{pmatrix} a & b \\ -b & d \end{pmatrix}, \quad E = E'_\pm(a, b, d) = \frac{1}{2} \left[a + d \pm \sqrt{(a - d)^2 - 4b^2} \right].$$

It can *still* be considered Hermitian (or, in the language of the review paper [5], quasi-Hermitian) *after* one redefines the Hilbert space accordingly (cf. [14] for more details).

In the language of phenomenology, one notices an important complementarity between the parameter-dependence of the two toy spectra $E_{\pm}(a, b, d)$ and $E'_{\pm}(a, b, d)$. In the “classical”, former example, *all* of the energies $E_{\pm}(a, b, d)$ remain safely real. The second, primed model is less easy to deal with. There exists the whole set of the eligible two-by-two metric operators $\Theta = \Theta^{\dagger} > 0$ which define the inner product in the corresponding two-dimensional toy Hilbert space \mathcal{H}' (cf. [15]). Thus, in spite of its manifest non-Hermiticity in the auxiliary two-dimensional Hilbert space \mathcal{H} (where the metric is the Dirac’s simplest identity operator), the operator $H'(a, b, d)$ represents an observable and remains safely compatible with all the postulates of Quantum Mechanics (cf. reviews [5, 12] for more details).

For $H = H(a, b, d) = H^{\dagger}$ the three-dimensional physical domain $\mathcal{D}(a, b, d)$ of parameters giving real spectra coincides with *all* \mathbb{R}^3 . In contrast, for each individual choice of the parameters a , b and d , the quasi-Hermiticity property of the primed Hamiltonian $H'(a, b, d)$ must be guaranteed and proved. In general, the reality of the bound-state energies $E'_{\pm}(a, b, d)$ and/or the stability of the primed system can only be achieved inside a *perceivably smaller* domain $\mathcal{D}'(a, b, d)$ with the easily specified EP horizon,

$$\partial\mathcal{D}'(a, b, d) = \left\{ (a, b, d) \in \mathbb{R}^3 \mid (a - d)^2 = 4b^2 \right\}.$$

Thus, the interior of the non-compact manifold $\mathcal{D}'(a, b, d)$ is specified by the single elementary constraint $b \in (-|a - d|, |a - d|)$.

Of course, for *qualitative* considerations the variability of parameters a and d is entirely redundant. It makes sense to get rid of them by the multiplicative re-scaling of all the parameters and by the subsequent shift of the energy scale leading to the “generic” choice of $a = -1$ and $d = 1$. An extension of this argument has been formulated in refs. [16] – [19]. A support has been found there for the study of the very special family of matrix models $H^{(N)}$. In our present paper we just intend to add new results showing that and how the respective quasi-Hermiticity domains \mathcal{D} can be described, at the lowest dimensions, by non-numerical means.

2.2 Tridiagonal chain models

One of the most surprising byproducts of our studies [14] – [21] was an empirical observation that certain particularly simple (i.e., in the sense of regularity, “canonical”) shapes of the J –parametric quasi-Hermiticity domains $\mathcal{D}^{(N)}$ can be found for a class of the generic, maximally simplified pseudo-Hermitian Hamiltonians $H^{(N)}$ chosen in

the following N -dimensional and tridiagonal “self-dual” [23] matrix form

$$H^{(N)} = \begin{bmatrix} -(N-1) & g_1 & & & & \\ & -g_1 & -(N-3) & g_2 & & \\ & & -g_2 & \ddots & \ddots & \\ & & & \ddots & N-5 & g_2 \\ & & & & -g_2 & N-3 & g_1 \\ & & & & & -g_1 & N-1 \end{bmatrix} \quad (1)$$

with a J -plet of real couplings $\vec{\lambda} = (g_1, g_2, \dots, g_J)$ and with the dimensions $N = 2J$ or $N = 2J + 1$. In a more explicit formulation, at any dimension N we found the coordinates of all the maximal-coupling spikes of the horizon $\partial\mathcal{D}^{(N)}$ in closed form,

$$g_n^{(spike)} = \pm(N-n)n, \quad n = 1, 2, \dots, J. \quad (2)$$

Although this result looks easy, its derivation from the underlying algebraic equations required extensive computer-assisted symbolic manipulations and nontrivial extrapolation guesswork [18]. Moreover, this closed-form description of the positions of the protruded spikes of the horizon $\partial\mathcal{D}^{(N)}$ (called, in [18], “extremely exceptional” points, EEPs) has been complemented, in our subsequent paper [19], by the strong-coupling description of $\partial\mathcal{D}^{(N)}$ based on perturbation ansatz

$$g_n = g_n^{(spike)} \sqrt{(1 - \gamma_n(t))}, \quad \gamma_n(t) = t + t^2 + \dots + t^{J-1} + G_n t^J. \quad (3)$$

This formula extrapolated, to all J , the rigorous $J \leq 2$ fine-tuning rules as derived in refs. [14, 18]. *A posteriori*, using sufficiently small “redundant” parameters $t \ll 1$, it proved valid in nonempty open vicinities of all the EEP vertices.

At the larger ts , i.e., far from the EEP spikes, the determination of the physical horizons $\partial\mathcal{D}^{(N)}$ of our models $H^{(N)}$ with J free real parameters becomes a more or less purely numerical task at the higher J s [20]. Up to now, non-numerical exceptions with $N = 2$ and $N = 3$ have been reported in [14] (where the very easy localization of the one-parametric interval $\mathcal{D}^{(2)} \equiv (-1, 1)$ of g_1 has been made) and in [18] (mentioning the very similar result $\mathcal{D}^{(3)} \equiv (-\sqrt{2}, \sqrt{2})$). For the next two dimensions $N = 4$ and $N = 5$ with two parameters, the explicit construction of the planar curves $\partial\mathcal{D}^{(N)}$ can be also found in ref. [18]. In what follows we intend to complement and extend these observations beyond $J = 2$ and to show that the closed-form constructions of the prototype horizons $\partial\mathcal{D}^{(N)}$ remain feasible up to the dimension as high as $N = 11$.

2.3 Secular equations

Once we choose $N = 2J$ or $N = 2J + 1$, abbreviate $E^2 = s$ and, at all the odd dimensions $N = 2J + 1$, ignore the persistent and trivial “middle” energy level $E_J^{(2J+1)} = 0$, we find out [18] that all the secular equations $\det(H^{(N)} - E) = 0$ have the same polynomial form,

$$s^J - \binom{J}{1} s^{J-1} P + \binom{J}{2} s^{J-2} Q - \binom{J}{3} s^{J-3} R + \dots = 0. \quad (4)$$

At all J and N , the coefficients P, Q, R, \dots are real polynomial functions of the squares g_k^2 of the J -plets of our real matrix elements. Once all the energies are assumed real (i.e., equivalently, once all the roots s_k of eq. (4) happen to be non-negative), we immediately deduce the following relations tractable as necessary conditions imposed upon our coefficients in (4),

$$\begin{aligned} \binom{J}{1} \cdot P &= s_1 + s_2 + \dots + s_J \geq 0, \\ \binom{J}{2} \cdot Q &= s_1 s_2 + s_1 s_3 + \dots + s_1 s_J + s_2 s_3 + s_2 s_4 + \dots + s_{J-1} s_J \geq 0, \\ \binom{J}{3} \cdot R &= s_1 s_2 s_3 + s_1 s_2 s_4 + \dots + s_{J-2} s_{J-1} s_J \geq 0, \\ &\dots \end{aligned} \quad (5)$$

In the opposite direction, the set of the necessary inequalities $P \geq 0, Q \geq 0, \dots$ is incomplete as it does not provide the desirable sufficient condition of quasi-Hermiticity. It admits complex roots s in general (take a sample secular polynomial $(s^2 + 1)(s - 2)$ for illustration).

3 Domains $\mathcal{D}^{(2J)}$ and $\mathcal{D}^{(2J+1)}$

Obviously, for a given prototype Hamiltonian $H^{(N)}$ and under the constraints (5), the determination of the quasi-Hermiticity domain $\mathcal{D}^{(N)} = \mathcal{D}(H^{(N)})$ is *equivalent* to the guarantee of the non-negativity of all the J roots s_k of eq. (4). The explicit forms of the corresponding sufficient conditions will now be given for the first ten smallest matrix dimensions $N = 2, 3, \dots, 11$.

3.1 Non-negativity of the root of eq. (4) at $J = 1$

At $J = 1$ the linear version $s - P = 0$ of secular eq. (4) has the single root $s_0 = P$. The non-negativity of this root is equivalent to the non-negativity of the coefficient P . This means that in terms of the single coupling $g_1 = a$ available at $J = 1$, the necessary and sufficient criteria of the observability of $H^{(2)} = H^{(2)}(a)$ or $H^{(3)} = H^{(3)}(a)$ read $P^{(2)}(a) = 1 - a^2 \geq 0$ and $P^{(3)}(a) = 4 - 2a^2 \geq 0$, respectively. Thus, in a way transferable, *mutatis mutandis*, to any dimension, the explicit definitions $\mathcal{D}^{(2)}(a) = (-1, 1)$ and $\mathcal{D}^{(3)}(a) = (-\sqrt{2}, \sqrt{2})$ of the quasi-Hermiticity domains may be re-read as definitions of the corresponding EP horizons $\partial\mathcal{D}^{(2)}(a) = \{-1, 1\}$ and $\partial\mathcal{D}^{(3)}(a) = \{-\sqrt{2}, \sqrt{2}\}$.

3.2 Non-negativity of all the roots of eq. (4) at $J = 2$

At $J = 2$ the quadratic version $s^2 - 2Ps + Q = 0$ of secular eq. (4) has two roots $s_{\pm} = P \pm \sqrt{P^2 - Q}$. These two roots remain real if and only if $B \equiv P^2 - Q \geq 0$. In the subdomain of parameters where $B \geq 0$ they remain both non-negative if and only if $P \geq 0$ and $Q \geq 0$. We can summarize that the required sufficient criterion reads

$$P \geq 0, \quad P^2 \geq Q \geq 0. \quad (6)$$

In an alternative approach, *without* an explicit reference to the available formula for s_{\pm} , let us contemplate the parabolic curve $y(s) = s^2 - 2Ps$ which remains safely positive, in the light of our assumption (5), at all the negative $s < 0$. This means that this curve can only intersect the horizontal line $z(s) = -Q$ at some non-negative points $s \geq 0$.

In this way the proof of non-negativity of all the roots of our secular equation degenerates to the proof that there exist two real points of intersection of the $J = 2$ parabola $y(s)$ with the horizontal line $z(s)$ (which lies below zero) at some $s \geq 0$. Towards this end we consider the minimum of the curve $y(s)$ which lies at the point s_0 such that $y'(s_0) = 0$, i.e., at $s_0 = P$. This minimum must lie *below* (or, at worst, at) the horizontal line of $z(s) = -Q \leq 0$. But the minimum value of $y(s_0)$ is known, $y(P) = -P^2$. Thus, the condition of intersection $y(s_0) \leq z(s_0)$ gives the formula $P^2 \geq Q$. QED.

Marginally, it is amusing to notice that once eq. (5) holds, the inequality $P^2 - Q \geq 0$ is equivalent to the reality of the roots simply because $P^2 - Q \equiv (s_1 - s_2)^2/4$. In

fact, even for some other two-parametric matrices, precisely this type of requirement is responsible for an important part of the EP boundary $\partial\mathcal{D}$ (cf. refs. [16, 17] for details).

3.3 Non-negativity of all the roots of eq. (4) at $J = 3$

Neither at $N = 6$ nor at $N = 7$ the sufficient condition of non-negativity of all the energy roots s is provided by the three necessary rules $P \geq 0$, $Q \geq 0$ and $R \geq 0$ of eq. (5). Let us return, therefore, to the second method used in paragraph 3.2 and derive another, “missing” inequality needed as a guarantee of the reality of the energies. In the first step one notices again that all the three components of the polynomial

$$y(s) = s^3 - 3Ps^2 + 3Qs = R, \quad J = 3$$

remain safely non-positive at $s < 0$. Whenever the roots are guaranteed real, their non-negativity $s_n \geq 0$ with $n = 1, 2, 3$ is already a consequence of the three constraints (5). The necessary condition of their reality is less trivial but it still can be deduced from the shape of the function $y(s)$ on the half-axis $s \geq 0$, i.e., from the existence and properties of a real maximum of $y(s)$ (at $s = s_-$) and of its subsequent minimum (at $s = s_+$). At both these points the derivative $y'(s) = 3s^2 - 6Ps + 3Q$ vanishes so that both the roots $s_{\pm} = P \pm \sqrt{P^2 - Q}$ of $y'(s)$ must be real and non-negative. This condition is always satisfied for the *real* roots s_k of $y(s)$ since

$$B = P^2 - Q \equiv \frac{1}{54} \left[(s_1 + s_2 - 2s_3)^2 + (s_2 + s_3 - 2s_1)^2 + (s_3 + s_1 - 2s_2)^2 \right] \geq 0.$$

In the next step, the necessary guarantee of the reality of the roots s_k will be understood again as equivalent to the doublet of the inequalities $y(s_-) \geq R$ and $y(s_+) \leq R$. Here we may insert $s_{\pm}^2 = 2Ps_{\pm} - Q$ and get the two inequalities which are more explicit,

$$2(P^2 - Q)s_- \leq PQ - R \leq 2(P^2 - Q)s_+. \quad (7)$$

They restrict the range of a new symmetric function of the roots,

$$PQ - R \equiv \frac{1}{9} [s_1s_2(s_1 + s_2 - 2s_3) + s_2s_3(s_2 + s_3 - 2s_1) + s_3s_1(s_3 + s_1 - 2s_2)].$$

After another insertion of the known s_{\pm} we arrive at a particularly compact formula

$$2(P^2 - Q)^{3/2} \geq R - 3PQ + 2P^3 \geq -2(P^2 - Q)^{3/2}$$

or, equivalently,

$$4 \left(P^2 - Q \right)^3 \geq \left(R - 3 P Q + 2 P^3 \right)^2 .$$

Due to the numerous cancellations the latter relation further degenerates to the most compact missing necessary condition

$$3 P^2 Q^2 + 6 R P Q \geq 4 Q^3 + R^2 + 4 R P^3 . \quad (8)$$

Our task is completed. In combination with eqs. (5), equation (8) plays the role of the guarantee of the reality of the energy spectrum.

3.4 Non-negativity of all the roots of eq. (4) at $J = 4$

In a search for the non-negative roots of the quartic secular equation

$$\det \left(H^{(8,9)} - E I \right) = x^4 - 4 P x^3 + 6 Q x^2 - 4 R x + S \equiv y(x) + S = 0 \quad (9)$$

we note that all the four N -dependent coefficients P , Q , R and S again evaluate as certain polynomials in the squares of the four coupling parameters g_k , $k = 1, 2, 3, 4$. Once all these four expressions are kept non-negative, the curves $y(x)$ and $z(x) = -S$ do not intersect at $x < 0$. At $x \geq 0$ they do intersect four times at $x \geq 0$ (as required), provided only that the three extremes of $y(x)$ can be found at the three non-negative real roots $x_{1,2,3}$ of the extremes-determining equation

$$y'(x_{1,2,3}) = 4 (x_{1,2,3}^3 - 3 P x_{1,2,3}^2 + 3 Q x_{1,2,3} - R) = 0 . \quad (10)$$

In an ordering $0 \leq x_1 \leq x_2 \leq x_3$ of these roots we arrive at the three sufficient conditions

$$y(x_1) \leq -S, \quad y(x_2) \geq -S, \quad y(x_3) \leq -S \quad (11)$$

guaranteeing that the parameters lie inside $\mathcal{D}^{(8)}$ or $\mathcal{D}^{(9)}$.

All the three quantities x_k satisfy the cubic equation $y'(x) = 0$ so that its pre-multiplication by x enables us to eliminate the three fourth powers from $y(x)$,

$$x_{1,2,3}^4 = 3 P x_{1,2,3}^3 - 3 Q x_{1,2,3}^2 + R x_{1,2,3} .$$

Their insertion reduces all the three items of eq. (11) to the other three intermediate polynomial inequalities of the third degree,

$$-P x_{1,3}^3 + 3 Q x_{1,3}^2 - 3 R x_{1,3} + S \leq 0 ,$$

$$-P x_2^3 + 3 Q x_2^2 - 3 R x_2 + S \geq 0.$$

Repeating the same trick once more, the elimination of $x_{1,2,3}^3 = 3 P x_{1,2,3}^2 - 3 Q x_{1,2,3} + R$ gives an equivalent triplet of inequalities

$$-B x_1^2 + 2 B^{3/2} C x_1 \leq B^2 D, \quad (12)$$

$$-B x_2^2 + 2 B^{3/2} C x_2 \geq B^2 D, \quad (13)$$

$$-B x_3^2 + 2 B^{3/2} C x_3 \leq B^2 D. \quad (14)$$

Here, the old abbreviations $B = P^2 - Q$ and $2 B^{3/2} C = PQ - R$ plus a new one, $3 B^2 D = PR - S$ enable us to re-scale $x_{1,2,3} = \sqrt{B} Y_{1,2,3}$ which yields our final triplet of quadratic-equation conditions

$$Y_1^2 - 2 C Y_1 + D \geq 0, \quad (15)$$

$$Y_2^2 - 2 C Y_2 + D \leq 0, \quad (16)$$

$$Y_3^2 - 2 C Y_3 + D \geq 0. \quad (17)$$

The auxiliary roots $Y_{\pm} = C \pm \sqrt{C^2 - D}$ must be real and non-negative. In this way we must guarantee that $D \geq 0$ and $C^2 \geq D$. The conclusion is that eqs. (15) – (17) degenerate to the four final elementary requirements

$$Y_1 \leq Y_- \leq Y_2 \leq Y_+ \leq Y_3. \quad (18)$$

They complement the inequalities $B \geq 0$, $Q \geq 0$ and $-1 \leq C - \sqrt{1 + Q/B} \leq 1$ and form the complete algebraic definition of the domains $\mathcal{D}^{(8)}$ and $\mathcal{D}^{(9)}$.

3.5 Non-negativity of all the roots of eq. (4) at $J = 5$

Let us finally proceed to $H^{(N)}$ with $N = 10$ and/or $N = 11$ which leads to the “unsolvable” secular equations of the fifth degree,

$$x^5 - 5 P x^4 + 10 Q x^3 - 10 R x^2 + 5 S x - T \equiv y(x) - T = 0. \quad (19)$$

From our present point of view the problem of the construction of the respective horizons $\partial \mathcal{D}^{(N)}$ remains solvable exactly since the derivative $y'(x)$ is still of the mere fourth degree,

$$\frac{1}{5} y'(x) = x^4 - 4 P x^3 + 6 Q x^2 - 4 R x + S. \quad (20)$$

The exact, real and non-negative values $x_1 \leq x_2 \leq x_3 \leq x_4$ of the four roots of $y'(x)$ may still be considered available in closed form.

In a way which parallels our preceding considerations we may assume that the five N -dependent non-negative coefficients $P \geq 0$, $Q \geq 0$, $R \geq 0$, $S \geq 0$ and $T \geq 0$ obey also all the additional inequalities derived in the preceding sections. In a more detailed description, we may then treat our secular problem (19) as a search for the graphical intersections between the (nonnegative) constant curve $z(x) = T$ and the graph of the polynomial $y(x)$ of the fifth degree (which can only be nonnegative at $x \geq 0$). Inside the domain $\mathcal{D}^{(N)}$, the quintuplet of the (unknown but real and nonnegative) physical energy roots x_a , x_b , x_c , x_d and x_e may be assumed compatible with the obvious intertwining rule

$$0 \leq x_a \leq x_1 \leq x_b \leq x_2 \leq x_c \leq x_3 \leq x_d \leq x_4 \leq x_e.$$

The way towards the sufficient condition of the existence of the real energy spectrum remains the same as above, requiring

$$y(x_1) \geq T, \quad y(x_2) \leq T, \quad y(x_3) \geq T, \quad y(x_4) \leq T. \quad (21)$$

The lowering of the degree should again reduce eq. (21) to the quadruplet

$$w(Y_1) \leq 0, \quad w(Y_2) \geq 0, \quad w(Y_3) \leq 0, \quad w(Y_4) \geq 0. \quad (22)$$

where the re-scaling $x_{1,2,3,4} = Y_{1,2,3,4} \sqrt{B}$ applies to the arguments of the brand new auxiliary polynomial function of the third degree in Y ,

$$w(Y) = Y^3 - 3CY^2 + 3DY - G.$$

Besides the same abbreviations as above, we introduced here a new one, for $PS - T \equiv 4B^{5/2}G$. The new and specific problem now arises in connection with the necessity of finding the three auxiliary and, of course, real and non-negative roots of the cubic polynomial $w(Y)$. Once we mark them, in the ascending order, by the Greek-alphabet subscripts, we should either postulate our (in principle, explicit) knowledge of their real and nonnegative values $Y_\alpha \leq Y_\beta \leq Y_\gamma$ or, in another perspective, we have to add the above-studied conditions which restrict the range of the three coefficients C , D and G in the cubic polynomial $w(Y)$.

This being said, the rest of the story is easy to tell. Once we parallel the same geometric argument as used in our previous subsections, we may immediately conclude that our “last feasible” specification of the domains $\mathcal{D}^{(10)}$ and $\mathcal{D}^{(11)}$ will be given by the following set of the inequalities,

$$Y_1 \leq Y_\alpha \leq Y_2 \leq Y_\beta \leq Y_3 \leq Y_\gamma \leq Y_4. \quad (23)$$

This is the last algebraic formula which defines the domains $\mathcal{D}^{(10)}$ and $\mathcal{D}^{(11)}$. Any extension of the recipe beyond $N = 11$ would suffer from the necessity of using certain purely numerically defined auxiliary functions of couplings g_k .

4 Discussion

4.1 Reparametrizing the couplings

The existence of the algebraic formulae which determine all the boundaries $\partial\mathcal{D}^{(N)}$ up to $N = 11$ opens a way towards a non-perturbative extension of the results of refs. [18] and [19] beyond the strong-coupling dynamical regime. In such a setting, the old perturbation ansatz (3) can be re-interpreted as a *precise*, non-perturbative change of variables $g_k \longrightarrow G_k$. During such a process, the redundant value of t may be fixed arbitrarily (cf. the construction of the planar curve $\partial\mathcal{D}^{(4)}$ in section 3.1 of paper [19] for illustration). Due to the reflection symmetries of our hedgehog-shaped horizons $\partial\mathcal{D}^{(N)}$, one is also allowed, without any loss of generality, to restrict attention to the subdomain of $\mathcal{D}^{(N)}$ with positive g_k s. In such a setting the new, rescaled real couplings $\gamma_J^{(N)} = \alpha$, $\gamma_{J-1}^{(N)} = \beta$, ... should remain non-negative and smaller than one, $\gamma_k^{(N)} \in (0, 1)$. For illustration one may recollect the most elementary two-by-two Hamiltonian

$$H^{(2)} = \begin{pmatrix} -1 & \sqrt{1-\alpha} \\ -\sqrt{1-\alpha} & 1 \end{pmatrix}, \quad \alpha \in (0, 1) \quad (24)$$

with the two-point spectrum $E_{\pm}^{(2)} = \pm\sqrt{\alpha}$. With respect to the new parameter we have to set $\mathcal{D}^{(2)}(\alpha) \equiv (0, 1)$ since there are no additional constraints.

4.2 New forms of approximations

We should emphasize that in comparison with our previous work, the most significant progress has been achieved here in the new non-approximate and complete description of the structure of the boundaries of the domains $\mathcal{D}^{(N)}$ at $N = 6$ and $N = 7$ by means of our key inequality (7). In order to stress some of its merits, let us now add a few comments on this form of the rigorous guarantee of the reality of the energies at $J = 3$.

Firstly, let us set $P^2 = B + Q$ and switch just to the postulates $B \geq 0$ and $Q \geq 0$, re-classifying the expression $P = +\sqrt{B + Q}$ itself as a mere formal abbreviation.

This enables us to insert

$$s_{\pm} = \sqrt{B+Q} \pm \sqrt{B} \geq 0$$

in eq. (7) which prescribes $s_- \leq C \sqrt{B} \leq s_+$ with $PQ - R = 2 B^{3/2} C \geq 0$, i.e.,

$$\sqrt{1+q} - 1 \leq C \leq \sqrt{1+q} + 1, \quad q = \frac{Q}{B} \in (0, \infty). \quad (25)$$

Once we notice that $PQ \equiv Q \sqrt{B+Q} = q B^{3/2} \sqrt{1+q}$ we may return from the auxiliary C to the original R and rewrite eq. (25) as a perceivably simplified one-parametric constraint

$$1 + \left(\frac{q}{2} - 1\right) \sqrt{1+q} \geq \frac{R}{2 B^{3/2}} \geq \begin{cases} 0, & q \leq 3, \\ \left(\frac{q}{2} - 1\right) \sqrt{1+q} - 1, & q > 3 \end{cases} \quad (26)$$

imposed upon the rescaled polynomial R . It completes our specification of the physical domain $\mathcal{D}^{(6,7)}$ in terms of the coefficients $B \geq 0$, $Q \geq 0$ and R . We see that with

$$1 + \left(\frac{q}{2} - 1\right) \sqrt{1+q} = \frac{3}{8}q^2 - \frac{1}{8}q^3 + \frac{9}{128}q^4 - \frac{3}{64}q^5 + \frac{35}{1024}q^6 + \mathcal{O}(q^7),$$

the two-sided inequality (26) is fairly restrictive in the R -direction, especially at the smallest ratios $q = Q/B$.

4.3 Pairwise confluences of the levels

Several aspects of the “first nontrivial” $J = 3$ problem have been skipped in the main text but they definitely deserve more attention in the discussion. For the sake of definiteness let us choose just $N = 6$ and use

$$g_1 = c = \sqrt{5(1-\gamma)}, \quad g_2 = b = 2\sqrt{2(1-\beta)}, \quad g_3 = a = 3\sqrt{1-\alpha}$$

with parameters $\alpha, \beta, \gamma \in (0, 1)$ entering the six-by-six matrix (1). Its secular equation (4) determines the spectrum which remains real, by definition, whenever all the couplings lie inside the domain $\mathcal{D}^{(6)}$. It is easy to deduce that the latter domain is circumscribed by the ellipsoidal surface given by the equation

$$P = -(a^2 + 2b^2 + 2c^2 - 35)/3 = 0.$$

The other two obvious constraints read

$$3Q = b^4 + 2c^2a^2 - 44b^2 + 28c^2 - 34a^2 + c^4 + 259 + 2b^2c^2 \geq 0$$

and

$$-R = a^2 c^4 - 10 b^2 c^2 + 30 c^2 a^2 + 225 a^2 - 30 c^2 - c^4 - 25 b^4 - 225 - 150 b^2 \geq 0.$$

The last constraint needed to define $\mathcal{D}^{(6)}$ is then given by eq. (7). In its light one can spot certain new structures in $\partial\mathcal{D}^{(6)}$ where, for example, the total EEP confluence of all the energies can be preceded by some incomplete, pairwise coincidences among the six levels in question. For example, an “innermost” pair of the energies can coincide at $E = 0$ while, independently, the other two “outer” doublets of the energies are allowed to coincide at the two symmetric non-vanishing values $E = \pm 4z$ at an unknown parameter $z \in (0, 1)$. Alternatively, the two “outermost” levels [with $s = s_{max} = 5y$ where $y \in (0, 1)$] can stay real while the confluence only involves the two internal energy doublets at a shared value of $s = s_{min} = 4x$ where $x \in (0, 1)$.

In the latter scenario one has to reproduce the two-parametric relation

$$(s - s_{max}) [s - s_{min}]^2 = s^3 - (32x^2 + 25y^2)s^2 + (256x^4 + 800x^2y^2)s - 6400x^4y^2 = 0$$

which, implicitly, defines the third sub-surface. Let us concentrate on the former, slightly simpler scenario where the surface of the three-dimensional domain $\mathcal{D}^{(6)}$ can be visualized as composed, locally, of the two eligible smooth sub-surfaces which intersect along a certain “double exceptional point” (DEP) curve.

4.4 Pairwise confluences of exceptional points

In terms of the single free parameter z of the latter particular scenario, the DEP secular equation degenerates to the formula $E^2 (E + 4z)^2 (E - 4z)^2 = 0$, to be obtained, in the one-parametric DEP limit, from eq. (4), i.e.,

$$s [s - (4z)^2]^2 = s^3 - 2(4z)^2 s^2 + (4z)^4 s = 0. \quad (27)$$

It is fortunate that the necessary analysis can still be performed non-numerically since equation (27) is easy to compare with the true secular equation (4) with coefficients given in paragraph 4.3. As long as the factorizable coefficient at s^0 must vanish, we get the first DEP constraint

$$[ac^2 + 15a + (15 + c^2 + 5b^2)] [ac^2 + 15a - (15 + c^2 + 5b^2)] = 0$$

so that we may eliminate

$$a = \pm \frac{15 + c^2 + 5b^2}{c^2 + 15}.$$

In the quadrant of $a - b - c$ space with positive a the plus sign must be chosen,

$$a = 1 + \frac{5b^2}{c^2 + 15}$$

i.e., we have $3 \geq a \geq 1$ in the closed formula for

$$b^2 = \frac{1}{5} (c^2 + 15) (a - 1) \quad (28)$$

or, alternatively, for

$$c^2 = \frac{5b^2}{a - 1} - 15.$$

This result is to be complemented by the other two relations

$$3Q(c, b, a) = 32z^2, \quad R(c, b, a) = 128z^4.$$

A straightforward elimination of z^2 gives the second DEP condition

$$-66a^2 - 36b^2 + 4c^2a^2 - 189 + 252c^2 - 4b^2a^2 - a^4 = 0$$

with the two compact roots

$$a_{\pm}^2 = 2c^2 - 33 - 2b^2 \pm 2\sqrt{c^4 + 30c^2 - 2b^2c^2 + 225 + 24b^2 + b^4}.$$

The acceptable one must be non-negative. For a_-^2 this would mean that $2c^2 \geq 33 + 2b^2$ while, at the same time, $63 + 12b^2 \geq 84c^2$. These two conditions are manifestly incompatible so that we must accept the upper-sign root a_+^2 which is automatically positive for all the large $2c^2 \geq 33 + 2b^2$ and which remains positive for all the smaller c^2 constrained by the requirement

$$84c^2 \geq 63 + 12b^2.$$

After the insertion of the definition (28) of b^2 we arrive at the condition

$$84c^2 \geq 63 + \frac{12}{5} (a - 1) (15 + c^2) \quad (29)$$

with a non-empty domain of validity.

5 Conclusions

According to the abstract principles of Quantum Mechanics, observable quantities (say, the energies $E_0 < E_1 < \dots$ of bound states) should be constructed as eigenvalues of certain self-adjoint operators of observables (i.e., in our illustration, of a

Hamiltonian H acting in some physical Hilbert space of states \mathcal{H}). Of course, an explicit representation of \mathcal{H} can prove complicated. Hence, the idea emerged of an introduction of a simpler, “auxiliary” Hilbert space (to be denoted here as $\mathcal{H}^{(aux)}$).

Although the origin of the latter idea can be traced back to the very early days of Quantum Theory [13], the feasibility of its separate implementations have long been treated as a mere mathematical curiosity (cf., e.g., [24] for illustration). Among the people who felt forced to take it really very seriously were the nuclear-physics specialists. Incidentally, in their so called IBM models of atomic nuclei, one of the “natural” (often called Dyson’s) fermion-to-boson mappings $\mathcal{H} \leftrightarrow \mathcal{H}^{(aux)}$ happened to simplify the solution of the Schrödinger equation considerably. In 1992 the topic has nicely been reviewed by Scholtz et al [5].

In 1998, many other physicists of different professional orientations (ranging from supersymmetry [25] and field theory [26] to cosmology [27] etc) have got involved when Bender and Boettcher [28] attracted their attention to the specific class of examples

$$-\frac{\hbar^2}{2m} \frac{d^2}{dx^2} \psi^{(aux)}(x) + V^{(aux)}(x) \psi^{(aux)}(x) = E \psi^{(aux)}(x) \quad (30)$$

(cf. also refs. [29, 30]) where the potentials may be complex but where the spectra remain real [11]. Thus, although the physical space \mathcal{H} itself proves complicated (mainly, due to its highly nontrivial definition of the inner product [31]), the auxiliary space $\mathcal{H}^{(aux)}$ can often be chosen in its most common representation $\mathbb{L}_2(\mathbb{R})$ of the square-integrable complex functions of one variable. In this setting (cf. its recent review [12]), the difference between the clarity of the properties of the space of functions $\mathcal{H}^{(aux)} = \mathbb{L}_2(\mathbb{R})$ and the perceivably more complicated character of the physical states in \mathcal{H} emerges due to the nontriviality of the “physical metric” $\Theta \neq I$ in \mathcal{H} (cf. ref. [5] for a compact review of the necessary mathematical properties of this operator). In the context of eq. (30) (possessing also numerous exactly solvable special cases [30]), the metric Θ is assumed constructed as the product of certain operators \mathcal{P} (= parity) and \mathcal{C} (= “charge”).

Beyond the specific class of the differential phenomenological models (30), another obvious choice of $\mathcal{H}^{(aux)}$ would be finite-dimensional [5, 32]. One of the key merits of such an alternative extension of the class of the “tractable” models of bound states lies in the fact that the key mathematical proofs of the reality of their spectra may become decisively simpler. It is in the latter context, with $\dim \mathcal{H}^{(aux)} = N < \infty$, where our attention has been attracted to the various specific models where one can

extract more information about the *shape* of the spectral-reality domain \mathcal{D} of variable parameters in the Hamiltonians.

As we emphasized in [20], a nontrivial relationship may exist between the matrix, finite-dimensional models and the so called quantum catastrophes interpreted as changes of some parameters λ_j , $j = 1, 2, \dots, J$ in the Hamiltonians $H(\vec{\lambda})$ which lead to the loss of the reality (i.e., of the observability) of certain energies $E_{n_c}(\vec{\lambda})$ in the spectrum. In a step towards a typology of such a phenomenon we showed here that for our \mathcal{PT} -symmetric chain models $H^{(N)}$ of dimensions $N = 2, 3, \dots, 11$ a non-numerical knowledge becomes available about the parametric domains $\mathcal{D}^{(N)}$ in which all the energies remain real. This means that the “admissible” J -plets of the coupling constants have been shown defined by certain non-numerical means. In this way, the explicit control of the stability of the system becomes mediated via purely algebraic constraints imposed upon the controllable parameters.

In the conclusion we feel encouraged to express our belief that *many* qualitative and geometric features of the observability horizons $\partial\mathcal{D}$ assigned to *any given* \mathcal{PT} -symmetric Hamiltonian H may be expected to *survive* its reduction to a series of N by N approximations of the prototype form $H^{(N)}$. Such a feature would really enhance the relevance of our present study of the peculiar self-dual \mathcal{PT} -symmetric Hamiltonians $H^{(N)} \neq \left(H^{(N)}\right)^\dagger$. In particular, it would also assign a deeper meaning to our present detailed analysis of the boundaries $\partial\mathcal{D}^{(N)}$ at the first few dimensions up to $N = 11$. In such a setting and perspective it is also appropriate to re-emphasize the two reasons of the relevance of our knowledge of the physical domains \mathcal{D} . Firstly, their boundaries $\partial\mathcal{D}$ are marking the breakdown of the reality and observability of the spectrum $\{E_n\}$. Secondly, all the vicinity of these boundaries also represents a region where the matrices $H^{(N)}$ cease to be diagonalizable. Thus, it is the *simultaneous* degeneracy of the energies and of the wave functions which gives the full and deep physical meaning to this horizon of the dynamical stability of the system.

Acknowledgement

Work supported by the GAČR grant Nr. 202/07/1307, by the MŠMT “Doppler Institute” project Nr. LC06002 and by the Institutional Research Plan AV0Z10480505.

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